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A VARIATIONAL FORMULATION OF THE PERTURBED MOTION PROBLEM FOR A VISCOELASTIC BODY

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Abstract—The analysis of perturbed motion is often very important for studying the progress of strain and stress in viscoelastic bodies. The authors intend to provide a variational formulation of the problem as an alternative to the differential formulation used to date, by solving the so-called inverse problem of the calculus of variations. This paper shows how the operator ruling the problem can be made symmetric by using a convolution bilinear form to obtain four functionals which are stationary at the solution of the differential problem. In conclusion, for example, the two-dimensional equations of the perturbed motion of a viscoelastic thin plate, are derived from the stationary condition of the three-dimensional functional.

1. INTRODUCTION

The stability of motion of a viscoelastic body is usually analysed by studying the perturbed motion following a variation of data (disturbance). The problem, formulated under the assumptions of small displacements and disturbances, consists of a system of linear integro-differential equations (Bolotin, 1969), expressing the local equilibrium conditions at every instant, and related boundary and initial conditions. This formulation usually involves a number of problems.

In many structural problems (e.g. rods, plates or shells), the three-dimensional equations of the continuum are not dealt with but it is often preferred to simplify the unknown fields by introducing suitable constraints or by choosing to neglect some terms regarded as less meaningful. In this case, writing the field equations, especially the boundary conditions, may be particularly difficult if one starts from the strong formulation previously described, while it becomes much easier if one starts from a variational formulation, as demonstrated in elastic problems where a variational formulation related to the total energy is available.

The second problem is related to the approximation of the solution. This can be sought by means of step-by-step integration in time or numerical Laplace transforms. These methods are very demanding and, furthermore, the Laplace transform method can be applied only to the case of fundamental motion with stress constant in time. Even in this case, a variational formulation would be desirable, since it permits applying classical approximation methods (Dall'Asta and Menditto, 1993).

The authors therefore intend to deal with the so-called inverse problem of the calculus of variations by seeking one or more functionals which are stationary at the solution of the integro-differential equations describing the perturbed motion.

However the characterization of this functional is not a trivial matter and cannot be obtained in the classical way because the operator ruling the problem is not symmetric with respect to the classical bilinear form derived from the scalar product. The problems ruled by non-symmetric operators resisted variational formulation for a long time and the first results are due to Gurtin (1963, 1964) and to Morse and Fenshbach (1953), with reference to particular problems. A clearer outlook on the problem was finally obtained by Tonti (1973) who focused the question on the meaning of the operator symmetry and on the fact that this property depends on the particular bilinear form.

In this paper the authors investigate the possibility of obtaining a variational formulation by means of the convolution bilinear form. This bilinear form was previously used for the classical linear dynamic viscoelastic problem by Leitman (1966) whose success was principally due to the particular nature of the viscoelastic operator. In this case, the presence of stress state due to fundamental motion makes the considered operator more complex and substantially different. This creates some difficulty and it can be shown that symmetry is preserved only if the stress history satisfies a symmetry condition. However, this limitation can be by-passed considering a time interval which is twice as long and, for stress varying in time, the difficulties related to the Laplace transform method can be avoided.

In this paper four different variational formulations are proposed: three of these involve relaxation type viscoelastic material and one involves creep type viscoelastic material. The latter can be particularly useful in practical applications because the constitutive laws derived from experimental data are often expressd by means of creep laws.

In conclusion, as an application example, the problem of perturbed motion of thin plates is derived from the proposed variational formulation of the three-dimensional continuum. The usual Kirchhoff assumptions permit obtaining a two-dimensional problem and determining the boundary conditions, in which the coupling between the Kirchhoff shear force and fundamental motion stress is shown.

2. VARIATIONAL FORMULATION

Let Ω be a bounded Kellog-regular region occupied by a viscoelastic body in the three-dimensional Euclidean space, $\partial\Omega$ its boundary and $\bar{\Omega}$ its closure; let $[0, T]$ be a time interval. The displacement vector at the point $\mathbf{x} \in \bar{\Omega}$ at the instant $t \in [0, T]$ is denoted by $\mathbf{u}$ and the symmetric strain and stress tensors are denoted, respectively, by $\mathbf{E}$ and $\mathbf{S}$. It is assumed that the process $[\bar{\mathbf{u}}, \bar{\mathbf{E}}, \bar{\mathbf{S}}]$, solving the linear viscoelastic problem with the assumptions of small displacement and stain, is known.

The infinitesimal stability of the process is usually analysed by studying the perturbed process $[\tilde{\mathbf{u}}, \tilde{\mathbf{E}}, \tilde{\mathbf{S}}]$ with $\tilde{\mathbf{u}} = \bar{\mathbf{u}}+\mathbf{u}$, $\tilde{\mathbf{E}} = \bar{\mathbf{E}}+\mathbf{E}$, $\tilde{\mathbf{S}} = \bar{\mathbf{S}}+\mathbf{S}$, occurring in the presence of the data $\tilde{\mathbf{f}} = \bar{\mathbf{f}}+\mathbf{f}$ (forces per unit mass), $\tilde{\mathbf{g}} = \bar{\mathbf{g}}+\mathbf{g}$ (presented displacements on the boundary), $\tilde{\mathbf{h}} = \bar{\mathbf{h}}+\mathbf{h}$ (prescribed tractions on the boundary), $\tilde{\mathbf{u}}_0 = \bar{\mathbf{u}}_0+\mathbf{u}_0$ (initial displacements), $\tilde{\mathbf{v}}_0 = \bar{\mathbf{v}}_0+\mathbf{v}_0$ (initial velocities), and by linearizing the equilibrium equations, assuming that $\|\nabla\mathbf{u}\| \ll \|\nabla\bar{\mathbf{u}}\| \ll 1$ (Bolotin, 1964, 1969). The unknowns $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ can be determined by solving the differential problem governed by the following field equations:

$$\nabla_s\mathbf{u} - \mathbf{E} = 0 \quad \text{on } \Omega \times (0, T), \tag{1}$$

$$\mathbf{G} \circledast \mathbf{E} - \mathbf{S} = 0 \quad \text{on } \Omega \times (0, T), \tag{2a}$$

or, alternatively

$$\mathbf{J} \circledast \mathbf{S} - \mathbf{E} = 0 \quad \text{on } \Omega \times (0, T), \tag{2b}$$

and

$$-\operatorname{div}(\mathbf{S} + \nabla\mathbf{u}\bar{\mathbf{S}}) + \rho\ddot{\mathbf{u}} = \rho\mathbf{f} \quad \text{on } \Omega \times (0, T), \tag{3}$$

and by the boundary and initial conditions:

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega_u \times [0, T], \tag{4}$$

$$(\mathbf{S} + \nabla\mathbf{u}\bar{\mathbf{S}})\mathbf{n} = \mathbf{h} \quad \text{on } \partial\Omega_S \times [0, T], \tag{5}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{on } \bar{\Omega} \times t = 0, \tag{6}$$

$$\dot{\mathbf{u}}|_{t=0} = \mathbf{v}_0 \quad \text{on } \bar{\Omega} \times t = 0, \tag{7}$$

where ∇ denotes the gradient of a vector and ∇_s its symmetric part, $\mathbf{G}$ and $\mathbf{J}$ the four-order tensors describing the kernel of the viscoelastic constitutive behaviour mapping symmetric tensors into symmetric tensors, $\circledast$ the Boltzmann operator,† div the divergence operator, the superimposed dots the temporal derivatives, $\partial\Omega_u$ and $\partial\Omega_S$ two disjoint subsets of the boundary ($\partial\Omega_u \cap \partial\Omega_S = 0$) such that $\partial\Omega_u \cup \partial\Omega_S = \partial\Omega$, $\mathbf{n}$ the normal outward on $\partial\Omega$, ρ is a positive function describing the mass density; $\bar{\mathbf{f}}$, $\bar{\mathbf{g}}$, $\bar{\mathbf{h}}$, $\bar{\mathbf{u}}_0$, $\bar{\mathbf{v}}_0$ are assigned functions forming the problem data. If the constitutive law is defined by means of eqn (2a), the material is said to be viscoelastic of relaxation type, otherwise the material is viscoelastic of creep type, eqn (2b).

The classic problem is usually posed on spaces of continuous functions. In this paper it is more generally assumed that the data are square-integrable in their domain and the differential equations must be satisfied almost everywhere (a.e.). It is also assumed that the unknowns are defined in the following sets :‡ $\mathbf{u} \in U = H^2((0,T)\,;\,H^1(\Omega))$, $\mathbf{S} \in \Sigma = \{\mathbf{S} \in \mathbf{L}^2((0,T)\,;\,L^2(\Omega)) : \text{div}\, S \in L^2((0,T)\,;\,L^2(\Omega))\}$, and $\mathbf{E} \in \Sigma$.

This is a linear problem in the form:

$$\mathrm{L}a = b, \tag{8}$$

where $\mathrm{L} : D(\mathrm{L}) \to R(\mathrm{L})$, $D(\mathrm{L})$ is a subset of a Banach space A and $R(\mathrm{L})$ is dense in a second Banach space B.

The authors intend to deal with the so-called inverse problem of the calculus of variations, by seeking a functional $\mathscr{F}(a) : D(\mathrm{L}) \to \mathbb{R}$ which is stationary at the solution of the problem (8), i.e.

$$\delta\mathscr{F}(a\,;\delta a) = 0 \quad \forall\, \delta a \in A \Leftrightarrow \mathrm{L}a = b. \tag{9}$$

This is a classical topic of applied mathematics and a historical review can be found in Tonti (1984). In the linear case the following result can be shown: if a form $\langle a, b\rangle : A \times B \to \mathbb{R}$ is bilinear, continuous and non degenerate, i.e.

$$\langle a, b\rangle = 0 \quad \forall\, a \in A \Rightarrow b = 0, \tag{10a}$$

$$\langle a, b\rangle = 0 \quad \forall\, b \in B \Rightarrow a = 0, \tag{10b}$$

and if the linear operator L is symmetric in the sense that it satisfies the condition:

$$\langle a_2, \mathrm{L}a_1\rangle = \langle a_1, \mathrm{L}a_2\rangle \quad \forall\, a_1, a_2 \in A, \tag{11}$$

then the functional sought exists and possesses the following form

$$\mathscr{F}(a) = 1/2\langle a, \mathrm{L}a\rangle - \langle a, b\rangle. \tag{12}$$

The examined problem does not permit a variational formulation by means of the classical procedure in which the data space and the unknowns space are related through the usual bilinear form derived from the scalar product. In fact, for spaces of vector valued functions this form assumes the following expression:

† The Boltzmann operator is the linear operator characteristic of the viscoelastic constitutive law, as introduced by Leitman and Fisher (1973). This is defined by the following equality:

$$\mathbf{G}(t) \circledast \mathbf{E}(t) = \mathbf{G}(0)\mathbf{E}(t) + \dot{\mathbf{G}}(t) * \mathbf{E}(t) = \mathbf{G}(0)\mathbf{E}(t) + \int_0^t \dot{\mathbf{G}}(t-\tau)\mathbf{E}(\tau)\,\mathrm{d}\tau,$$

where the components of $\mathbf{G}$ are absolutely continuous functions.

‡ L^2 denotes spaces of square-integrable functions and H^n spaces of functions that possess partial derivative in L^2, up to the order n.

$$\langle \mathbf{a}, \mathbf{b} \rangle = \int_\Omega \int_0^T \mathbf{a}(\mathbf{x}, t) \cdot \mathbf{b}(\mathbf{x}, t)\, \mathrm{d}\Omega\, \mathrm{d}t, \tag{13}$$

and it can be shown that the operator ruling the perturbed motion problem does not satisfy the symmetry requirement of eqn (11). A variational formulation can be obtained by adopting a convolution bilinear form, under suitable assumptions. This bilinear form has the following expression:

$$\langle \mathbf{a}, \mathbf{b} \rangle = \int_\Omega \int_0^T \mathbf{a}(\mathbf{x}, t) \cdot \mathbf{b}(\mathbf{x}, T-t)\, \mathrm{d}\Omega\, \mathrm{d}t = \int_\Omega \mathbf{a} * \mathbf{b}\, \mathrm{d}\Omega, \tag{14}$$

and possesses the properties required in eqn (10) (Yosida, 1980) but, contrary to the scalar product, it is not positive.

Rather than demonstrating the symmetry relation of eqn (11) and building up the functional by means of eqn (12), the authors have preferred to demonstrate the result sought by following the classical procedure which consists in showing that the stationary condition of a particular functional are the same as the equations of the differential problem.

Theorem. The differential problems of eqns (1), (2a), (3), (4), (5), (6), (7) is assigned. If the tensor $\mathbf{G}(\mathbf{x}, t)$† is symmetric and $\bar{\mathbf{S}}(\mathbf{x}, t) = \bar{\mathbf{S}}(\mathbf{x}, T-t)$ a.e. on $\Omega \times (0, T)$ then the solutions of the differential problem coincide with the stationary points of the following functional:

$$\begin{aligned}\mathscr{F}(\mathbf{u}, \mathbf{E}, \mathbf{S}) = &\int_\Omega (\nabla_s \mathbf{u} - \mathbf{E}) * \mathbf{S}\, \mathrm{d}\Omega + \frac{1}{2} \int \mathbf{G} \circledast \mathbf{E} * \mathbf{E}\, \mathrm{d}\Omega + \frac{1}{2} \int_\Omega \nabla \mathbf{u} \bar{\mathbf{S}} * \nabla \mathbf{u}\, \mathrm{d}\Omega \\ &+ \frac{1}{2} \int_\Omega \rho \dot{\mathbf{u}} * \dot{\mathbf{u}}\, \mathrm{d}\Omega - \int_\Omega \rho \mathbf{f} * \mathbf{u}\, \mathrm{d}\Omega + \int_{\partial\Omega_u} (\mathbf{g} - \mathbf{u}) * (\mathbf{S} + \nabla \mathbf{u} \bar{\mathbf{S}})\mathbf{n}\, \mathrm{d}\partial\Omega_u - \int_{\partial\Omega_S} \mathbf{h} * \mathbf{u}\, \mathrm{d}\partial\Omega_S \\ &+ \int_\Omega \rho(\mathbf{u}|_{t=0} - \mathbf{u}_0) \cdot \dot{\mathbf{u}}|_{t=T}\, \mathrm{d}\Omega - \int_\Omega \rho \mathbf{v}_0 \cdot \mathbf{u}|_{t=T}\, \mathrm{d}\Omega. \end{aligned} \tag{15}$$

Proof. Part (a). It will be shown that a process $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ satisfying the differential problem makes the functional stationary. The differential with respect to a generic variation $[\delta\mathbf{u}, \delta\mathbf{E}, \delta\mathbf{S}]$ has the following form:

$$\begin{aligned}\delta\mathscr{F}(\mathbf{u}, \mathbf{E}, \mathbf{s}; \delta\mathbf{u}, \delta\mathbf{E}, \delta\mathbf{S}) = &\int_\Omega \nabla_s \delta\mathbf{u} * \mathbf{S}\, \mathrm{d}\Omega - \int_\Omega \delta\mathbf{E} * \mathbf{S}\, \mathrm{d}\Omega + \int_\Omega (\nabla_s \mathbf{u} - \mathbf{E}) * \delta\mathbf{S}\, \mathrm{d}\Omega \\ &+ \frac{1}{2} \int_\Omega \mathbf{G} \circledast \delta\mathbf{E} * \mathbf{E}\, \mathrm{d}\Omega + \frac{1}{2} \int_\Omega \mathbf{G} \circledast \mathbf{E} * \delta\mathbf{E}\, \mathrm{d}\Omega + \frac{1}{2} \int_\Omega \nabla \delta\mathbf{u} \bar{\mathbf{S}} * \nabla \mathbf{u}\, \mathrm{d}\Omega \\ &+ \frac{1}{2} \int_\Omega \nabla \mathbf{u} \bar{\mathbf{S}} * \nabla \delta\mathbf{u}\, \mathrm{d}\Omega + \int_\Omega \rho \dot{\mathbf{u}} * \delta\dot{\mathbf{u}}\, \mathrm{d}\Omega - \int_\Omega \rho \mathbf{f} * \delta\mathbf{u}\, \mathrm{d}\Omega \\ &+ \int_{\partial\Omega_u} \delta\mathbf{u} * (-\mathbf{S})\mathbf{n}\, \mathrm{d}\partial\Omega_u + \int_{\partial\Omega_u} \delta\mathbf{u} * (-\nabla \mathbf{u} \bar{\mathbf{S}})\mathbf{n}\, \mathrm{d}\partial\Omega_u + \int_{\partial\Omega_u} (\mathbf{u} - \mathbf{g}) * (-\delta\mathbf{S} - \nabla \delta\mathbf{u} \mathbf{S})\mathbf{n}\, \mathrm{d}\partial\Omega_u \\ &- \int_{\partial\Omega_S} \mathbf{h} * \delta\mathbf{u}\, \mathrm{d}\partial\Omega_S + \int_\Omega \rho \delta\mathbf{u}|_{t=0} \cdot \dot{\mathbf{u}}|_{t=T}\, \mathrm{d}\Omega + \int_\Omega \rho(\mathbf{u}|_{t=0} - \mathbf{u}_0) \cdot \delta\dot{\mathbf{u}}|_{t=T}\, \mathrm{d}\Omega - \int_\Omega \rho \mathbf{v}_0 \cdot \delta\mathbf{u}|_{t=T}\, \mathrm{d}\Omega. \end{aligned} \tag{16}$$

† $\mathbf{G}$ satisfies the relation $\mathbf{A} \cdot \mathbf{G}\mathbf{B} = \mathbf{B} \cdot \mathbf{G}\mathbf{A}$ for every symmetric tensor pair $\mathbf{A}, \mathbf{B}$, a.e. on $(0, T) \times \Omega$.

By means of a change in the integration variable and Heaviside function, the following can be demonstrated [see Tonti (1973) for the uni-dimensional case]:

$$\mathbf{G} \circledast \delta\mathbf{E} * \mathbf{E} = \mathbf{G} \circledast \mathbf{E} * \delta\mathbf{E}, \tag{17}$$

Thanks to the symmetry of $\bar{\mathbf{S}}$:

$$\nabla\mathbf{u} \cdot \nabla\delta\mathbf{u}\bar{\mathbf{S}} = \operatorname{tr}(\nabla\mathbf{u}\bar{\mathbf{S}}^{\mathrm{T}}\nabla^{\mathrm{T}}\delta\mathbf{u}) = \operatorname{tr}(\nabla\mathbf{u}\bar{\mathbf{S}}\nabla^{\mathrm{T}}\delta\mathbf{u}) = \nabla\mathbf{u}\bar{\mathbf{S}} \cdot \nabla\delta\mathbf{u}, \tag{18}$$

can be written (tr = trace) and, for the assumption $\bar{\mathbf{S}}(\mathbf{x}, t) = \bar{\mathbf{S}}(\mathbf{x}, T-t)$, the following equation holds:

$$\nabla\delta\mathbf{u}\bar{\mathbf{S}} * \nabla\mathbf{u} = \nabla\mathbf{u}\bar{\mathbf{S}} * \nabla\delta\mathbf{u}. \tag{19}$$

In conclusion, the regularity assumptions permit applying the Green formulas:

$$\int_\Omega \mathbf{S} * \nabla_s\delta\mathbf{u}\,\mathrm{d}\Omega = \int_\Omega -\operatorname{div}\mathbf{S} * \delta\mathbf{u}\,\mathrm{d}\Omega + \int_{\partial\Omega_u} \mathbf{Sn} * \delta\mathbf{u}\,\mathrm{d}\Omega + \int_{\partial\Omega_S} \mathbf{Sn} * \delta\mathbf{u}\,\mathrm{d}\Omega, \tag{20}$$

$$\int_\Omega \nabla\mathbf{u}\bar{\mathbf{S}} * \nabla\delta\mathbf{u}\,\mathrm{d}\Omega = \int_\Omega -\operatorname{div}(\nabla\mathbf{u}\bar{\mathbf{S}}) * \delta\mathbf{u}\,\mathrm{d}\Omega + \int_{\partial\Omega_u} \nabla\mathbf{u}\bar{\mathbf{S}}\mathbf{n} * \delta\mathbf{u}\,\mathrm{d}\partial\Omega_u + \int_{\partial\Omega_S} \nabla\mathbf{u}\bar{\mathbf{S}}\mathbf{n} * \delta\mathbf{u}\,\mathrm{d}\partial\Omega_S, \tag{21}$$

$$\int_\Omega \rho\dot{\mathbf{u}} * \delta\dot{\mathbf{u}}\,\mathrm{d}\Omega = \int_\Omega \rho\ddot{\mathbf{u}} * \delta\mathbf{u}\,\mathrm{d}\Omega + \int_\Omega \rho\dot{\mathbf{u}}|_{t=0} \cdot \delta\mathbf{u}|_{t=T}\,\mathrm{d}\Omega - \int_\Omega \rho\dot{\mathbf{u}}|_{t=T} \cdot \delta\mathbf{u}|_{t=0}\,\mathrm{d}\Omega, \tag{22}$$

and the differential can be rewritten in the following form:

$$\begin{aligned}
\delta\mathscr{F}(\mathbf{u},\mathbf{E},\mathbf{S};\delta\mathbf{u},\delta\mathbf{E},\delta\mathbf{S}) &= \int_\Omega (\nabla_s\mathbf{u}-\mathbf{E}) * \delta\mathbf{S}\,\mathrm{d}\Omega + \int_\Omega (\mathbf{G} \circledast \mathbf{E} - \mathbf{S}) \circledast \delta\mathbf{E}\,\mathrm{d}\Omega \\
&+ \int_\Omega (-\operatorname{div}(\mathbf{S}+\nabla\mathbf{u}\bar{\mathbf{S}}) + \rho\ddot{\mathbf{u}} - \rho\mathbf{f}) * \delta\mathbf{u}\,\mathrm{d}\Omega + \int_{\partial\Omega_u} (\mathbf{u}-\mathbf{g}) * (-\delta\mathbf{S} - \nabla\delta\mathbf{u}\bar{\mathbf{S}})\mathbf{n}\,\mathrm{d}\partial\Omega_u \\
&+ \int_{\partial\Omega_S} ((\mathbf{S}+\nabla\mathbf{u}\bar{\mathbf{S}})\mathbf{n} - \mathbf{h}) * \delta\mathbf{u}\,\mathrm{d}\partial\Omega_S + \int \rho(\mathbf{u}|_{t=0} - \mathbf{u}_0) \cdot \delta\dot{\mathbf{u}}|_{t=T}\,\mathrm{d}\Omega \\
&+ \int_\Omega \rho(\dot{\mathbf{u}}|_{t=0} - \mathbf{v}_0) \cdot \delta\mathbf{u}|_{t=T}\,\mathrm{d}\Omega,
\end{aligned} \tag{23}$$

which is null if $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ satisfies the differential problem.

Part (b). Vice-versa, if the functional is stationary at $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ then this process satifies the eqns (1), (2a), (3), (4), (5), (6), (7).

According to the form that the functional assumes in eqn (23) and because of the property of ρ and of the bilinear form [eqns (10)], it can be directly concluded that if the differential $\delta\mathscr{F}$ is null for every variation $[\delta\mathbf{u}, \delta\mathbf{E}, \delta\mathbf{S}]$ then the former terms in every integral must be null, i.e. eqns (1), (2a), (3), (4), (5), (6), (7) are satisfied. □

The proof is based on the two fundamental assumptions of symmetry for **G** and on the condition $\bar{\mathbf{S}}(\mathbf{x}, t) = \bar{\mathbf{S}}(\mathbf{x}, T-t)$.

The symmetry condition for the tensor **G** at every t is not thermodynamically necessary (Day, 1971; Fabrizio and Morro, 1988), but it is generally accepted in the formulation of viscoelastic problems (Gurtin and Sternberg, 1962; Gurtin, 1963; Leitman, 1966; Rionero and Chirita, 1989).

The condition $\bar{\mathbf{S}}(\mathbf{x}, t) = \bar{\mathbf{S}}(\mathbf{x}, T-t)$ is equivalent to requiring the symmetry around $T/2$ and this condition holds in the majority of engineering problems, in which the actions involving viscoelastic phenomena are constant. If $\bar{\mathbf{S}}(\mathbf{x}, t)$ does not vary symmetrically around $T/2$, the problem can be by-passed by studying the motion in the interval $[0, 2T]$ under a stress $\bar{\mathbf{S}}^*(\mathbf{x}, t)$ such that:

$$\bar{\mathbf{S}}^*(\mathbf{x}, t) = \bar{\mathbf{S}}(\mathbf{x}, t) \quad \text{on } \bar{\Omega} \times [0, T], \tag{24a}$$

$$\bar{\mathbf{S}}^*(\mathbf{x}, t) = \bar{\mathbf{S}}(\mathbf{x}, 2T-t) \quad \text{on } \bar{\Omega} \times (T, 2T]. \tag{24b}$$

Although the problem of existence is not analysed in this paper, it is useful to note that by following the above procedure, the problem studied differs from the primary problem. However, the existence of the transformed problem ensures the existence of the solution of the primary problem if the principle of determinism holds.

It must be observed that the bilinear form is not positive and the condition $\delta\mathscr{F} = 0$ simply denotes a stationary condition and not a minimum.

In the proposed formulation a general case has been considered but simplification is possible. In particular, in structural engineering topics it is often possible to neglect the dynamic terms involving velocities and initial conditions when the relaxation time of the material is much greater than the fundamental period of free-vibration of the structure (Hoff, 1958).

3. ALTERNATIVE VARIATIONAL FORMULATION

As in the elastic-static problem, $\mathscr{F}(\mathbf{u}, \mathbf{E}, \mathbf{S})$ can be referred to as the Hu–Washizu functional, since it depends on all the three unknown fields which vary independently from each other. Other functionals can be derived by introducing suitable constraints on the unknowns.

In fact, assuming that the tensor field $\mathbf{S}$ identically satisfies eqn (2a) ($\mathbf{G} \circledast \mathbf{E} - \mathbf{S} = 0$), the following functional can be obtained:

$$\begin{aligned}\mathscr{V}(\mathbf{u}, \mathbf{E}) = {} & \int_\Omega \nabla_s \mathbf{u} * \mathbf{G} \circledast \mathbf{E}\, d\Omega - \frac{1}{2}\int_\Omega \mathbf{G} \circledast \mathbf{E} * \mathbf{E}\, d\Omega + \int_{\partial\Omega_u} (\mathbf{g}-\mathbf{u}) * (\mathbf{G} \circledast \mathbf{E} + \nabla\mathbf{u}\bar{\mathbf{S}})\mathbf{n}\, d\partial\Omega_u \\ & + \frac{1}{2}\int_\Omega \nabla\mathbf{u}\bar{\mathbf{S}} * \nabla\mathbf{u}\, d\Omega + \frac{1}{2}\int_\Omega \rho\dot{\mathbf{u}} * \dot{\mathbf{u}}\, d\Omega - \int_\Omega \rho\mathbf{f} * \mathbf{u}\, d\Omega - \int_{\partial\Omega_S} \mathbf{h} * \mathbf{u}\, d\partial\Omega_S \\ & + \int_\Omega \rho(\mathbf{u}|_{t=0} - \mathbf{u}_0) \cdot \dot{\mathbf{u}}|_{t=T}\, d\Omega - \int_\Omega \rho\mathbf{v}_0 \cdot \mathbf{u}|_{t=T}\, d\Omega, \end{aligned} \tag{25}$$

or, vice-versa, assuming that $\mathbf{E}$ identically satisfies eqn (2b) ($\mathbf{J} \circledast \mathbf{S} - \mathbf{E} = 0$), another functional can be derived with the following expression:

$$\begin{aligned}\mathscr{T}(\mathbf{u}, \mathbf{S}) = {} & \int_\Omega \nabla_s \mathbf{u} * \mathbf{S}\, d\Omega - \frac{1}{2}\int_\Omega \mathbf{S} * \mathbf{J} \circledast \mathbf{S}\, d\Omega + \int_{\partial\Omega_u} (\mathbf{g}-\mathbf{u}) * (\mathbf{S} + \nabla\mathbf{u}\bar{\mathbf{S}})\mathbf{n}\, d\partial\Omega_u \\ & + \frac{1}{2}\int_\Omega \nabla\mathbf{u}\bar{\mathbf{S}} * \nabla\mathbf{u}\, d\Omega + \frac{1}{2}\int_\Omega \rho\dot{\mathbf{u}} * \dot{\mathbf{u}}\, d\Omega - \int_\Omega \rho\mathbf{f} * \mathbf{u}\, d\Omega - \int_{\partial\Omega_S} \mathbf{h} * \mathbf{u}\, d\partial\Omega_S \\ & + \int_\Omega \rho(\mathbf{u}|_{t=0} - \mathbf{u}_0) \cdot \dot{\mathbf{u}}|_{t=T}\, d\Omega - \int_\Omega \rho\mathbf{v}_0 \cdot \mathbf{u}|_{t=T}\, d\Omega. \end{aligned} \tag{26}$$

In this case $\mathscr{V}(\mathbf{u}, \mathbf{E})$ and $\mathscr{T}(\mathbf{u}, \mathbf{S})$ can be referred to as Hellinger–Reissner functionals. In particular, the latter is of remarkable operative interest since the viscoelastic constitutive laws derived from the experimental data are more frequently posed in the form of eqn (2b) rather than in the form of eqn (2a).

A functional similar to the total potential energy of the elastic case can be derived from eqn (25) by constraining **E** further so as to satisfy eqn (1), and constraining **u** in a subspace of functions satisfying the boundary condition of eqn (4) (**u** = **g**) on the boundary portion $\partial\Omega_u$. In the end, a functional defined for a generic field of admissible displacements **u**, is obtained in the following form:

$$\mathscr{E}(\mathbf{u}) = \frac{1}{2}\int_\Omega \mathbf{G} \circledast \nabla_s\mathbf{u} * \nabla_s\mathbf{u}\, d\Omega + \frac{1}{2}\int_\Omega \nabla\mathbf{u}\tilde{\mathbf{S}} * \nabla\mathbf{u}\, d\Omega + \frac{1}{2}\int_\Omega \rho\dot{\mathbf{u}} * \dot{\mathbf{u}}\, d\Omega$$

$$-\int_\Omega \rho\mathbf{f} * \mathbf{u}\, d\Omega - \int_{\partial\Omega_S} \mathbf{h} * \mathbf{u}\, d\partial\Omega_S + \int_\Omega \rho(\mathbf{u}|_{t=0} - \mathbf{u}_0)\cdot\dot{\mathbf{u}}|_{t=T}\, d\Omega - \int_\Omega \rho\mathbf{v}_0\cdot\mathbf{u}|_{t=T}\, d\Omega. \quad (27)$$

It must be remarked that only a formal analogy with total potential energy exists, because $\mathscr{E}(\mathbf{u})$ does not have a particular physical meaning and it is not extreme at the solution.

4. APPLICATION TO THIN PLATES

As an example of application, the field and boundary equations of perturbed motion of a thin plate are derived from the variational formulation of the three-dimensional continuum.

Let $\{\mathbf{e}_\alpha;\mathbf{e}_3\}$ be an orthonormal basis ($\alpha = 1,2$) and $(x_\alpha;x_3)$ the co-ordinate system. It is assumed that the spatial domain Ω, on which the problem is defined, coincides with the region $\Gamma\times(-h/2,h/2)$, where Γ denotes a regular domain on the plane $x_3 = 0$ with normal $\mathbf{n} = n_\alpha\mathbf{e}_\alpha$ and h the thickness. Dirichlet conditions are posed on $\partial\Omega_u = \partial\Gamma_u\times(-h/2,h/2)$ and Neumann conditions on $\partial\Omega_S = \partial\Gamma_S\times(-h/2,h/2)$.

Kirchhoff hypotheses are assumed; in the linear strain theory they are equivalent to assuming $S_{33} = \mathbf{S}\cdot\mathbf{e}_3\otimes\mathbf{e}_3 = 0$ and the internal constraints $E_{\alpha 3} = \mathbf{E}\cdot\mathbf{e}_\alpha\otimes\mathbf{e}_3 = 0$. The simpler constitutive law compatible with the assumed kinematic constraints is that of the transversally isotropic material and furnishes the active stress by means of the functions $\mu(t)$, $\lambda(t)$ and $\lambda'(t)$, leading to the following expressions (Podio-Guidugli, 1989):

$$\mathbf{S} = 2\mu \circledast \mathbf{E} + \lambda \circledast (\mathbf{E}\cdot\mathbf{I}_\alpha)\mathbf{I}_\alpha + \lambda' \circledast (\mathbf{E}\cdot\mathbf{I}_3)\mathbf{I}_3, \quad (28)$$

where $\mathbf{I}_\alpha$ denotes the metric tensor $\mathbf{e}_\alpha\otimes\mathbf{e}_\alpha$ and $\mathbf{I}_3$ denotes $\mathbf{e}_3\otimes\mathbf{e}_3$. The stress components $S_{\alpha 3} = \mathbf{S}\cdot\mathbf{e}_\alpha\otimes\mathbf{e}_3$ have a reactive nature and are not involved in the functional writing because their correspondent strain components $E_{\alpha 3}$ are null, while the condition S_{33} ensures that $E_{33} = \mathbf{E}\cdot\mathbf{e}_3\otimes\mathbf{e}_3 = 0$. This leads to the following displacement field **u**, for the considered geometry:

$$\mathbf{u} = (v_1 - x_3 w_{,1})\mathbf{e}_1 + (v_2 - x_3 w_{,2})\mathbf{e}_2 + w\mathbf{e}_3, \quad (29)$$

where v_α and w denote the displacement components of the middle surface with respect to the basis and are clearly functions of x_α only. The following expressions are obtained for the strain tensor:

$$\mathbf{E} = \nabla_s\mathbf{u} = (v_{1,1} - x_3 w_{,11})\mathbf{e}_1\otimes\mathbf{e}_1 + [1/2(v_{1,2}+v_{2,1}) - x_3 w_{,12}](\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1)$$
$$+ (v_{2,2} - x_3 w_{,22})\mathbf{e}_2\otimes\mathbf{e}_2, \quad (30)$$

and the displacement gradient:

$$\nabla\mathbf{u} = \nabla_s\mathbf{u} + 1/2(v_{2,1} - v_{1,2})(\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1) + w_{,1}(\mathbf{e}_3\otimes\mathbf{e}_1 - \mathbf{e}_1\otimes\mathbf{e}_3)$$
$$+ w_{,2}(\mathbf{e}_3\otimes\mathbf{e}_2 - \mathbf{e}_2\otimes\mathbf{e}_3). \quad (31)$$

The calculation is now carried out by considering only the last two terms of eqn (31), taking into account that, in stability problems, the rotations around $\mathbf{e}_\alpha$ prevail with respect to the rotation around $\mathbf{e}_3$ and the strain terms.

The case of fundamental motion producing only a plane state of stress ($\bar{S}_{\alpha 3} = \bar{S}_{33} = 0$), constant in thickness and in time, i.e. $\bar{\mathbf{S}} = \bar{\mathbf{S}}(x_1, x_2)$, is considered. It is assumed that the material characteristics can vary with respect to x_α, i.e. $\mu = \mu(x_\alpha; t)$ and $\lambda = \lambda(x_\alpha; t)$.

The displacements v_α, w can now be assumed as unknowns and the problem can be formulated by means of the functional $\mathscr{E}$ defined on admisible displacement fields, obtaining the following:

$$\mathscr{E}(v_\alpha, w) = \frac{1}{2}\int_\Gamma h[(2\mu+\lambda) \circledast (v_{1,1} * v_{1,1} + v_{2,2} * v_{2,2}) + 2\lambda \circledast v_{1,1} * v_{2,2}$$

$$+\mu \circledast (v_{1,2} * v_{1,2} + 2v_{1,2} * v_{2,1} + v_{2,1} * v_{2,1})]\,\mathrm{d}\Gamma$$

$$+\frac{1}{2}\int_\Gamma \frac{h^3}{12}[(2\mu+\lambda) \circledast (w_{,11} * w_{,11} + w_{,22} * w_{,22}) + 2\lambda \circledast (w_{,11} * w_{,22}) + 4\mu \circledast (w_{,12} * w_{,21})]\,\mathrm{d}\Gamma$$

$$+\frac{1}{2}\int_\Gamma h[\bar{S}_{11} w_{,1} * w_{,1} + 2\bar{S}_{12} w_{,1} * w_{,2} + \bar{S}_{22} w_{,2} * w_{,2}]\,\mathrm{d}\Gamma - \int_\Gamma h[f * w]\,\mathrm{d}\Gamma, \tag{32}$$

where the disturbance is formed by a force per unit volume $f(x_\alpha)\mathbf{e}_3$ constant vs x_3, the dynamic terms have been omitted and the integration on the thickness has been carried out. At this point the stationary condition $\delta\mathscr{E}(v_\alpha, w; \delta v_\alpha, \delta w) = 0$ can be used to solve the problem in its weak form.

The strong form of the problem can be derived from the stationary condition which, by applying the Green formulas, can be written as follows:

$$-\int_\Gamma h\{[(2\mu+\lambda) \circledast v_{1,1} + \lambda \circledast v_{2,2}]_{,1} + [\mu \circledast (v_{1,2} + v_{2,1})]_{,2}\} * \delta v_1\,\mathrm{d}\Gamma = 0 \quad \forall\, \delta v_1, \tag{33a}$$

$$-\int_\Gamma h\{[\mu \circledast (v_{2,1} + v_{1,2})]_{,1} + [(2\mu+\lambda) \circledast v_{2,2} + \lambda \circledast v_{1,1}]_{,2}\} * \delta v_2\,\mathrm{d}\Gamma = 0 \quad \forall\, \delta v_2, \tag{33b}$$

$$\int_{\partial\Gamma_S} h\{[(2\mu+\lambda) \circledast v_{1,1} + \lambda \circledast v_{2,2}]n_1 + [\mu \circledast (v_{1,2} + v_{2,1})]n_2\} * \delta v_1\,\mathrm{d}\partial\Gamma_s = 0 \quad \forall\, \delta v_1, \tag{33c}$$

$$\int_{\partial\Gamma_S} h\{[\mu \circledast (v_{2,1} + v_{1,2})]n_1 + [(2\mu+\lambda) \circledast v_{2,2} + \lambda \circledast v_{1,1}]n_2\} * \delta v_2\,\mathrm{d}\partial\Gamma_s = 0 \quad \forall\, \delta v_2, \tag{33d}$$

$$\int_\Gamma \left\{\frac{h^3}{12}[\Delta(\lambda \circledast \Delta w) + (2\mu \circledast w_{,11})_{,11} + (2\mu \circledast w_{,22})_{,22} + (4\mu \circledast w_{,12})_{,12}]\right.$$

$$\left. + h[(\bar{S}_{11} w_{,1})_{,1} + (\bar{S}_{12} w_{,1})_{,2} + (\bar{S}_{12} w_{,2})_{,1} + (\bar{S}_{22} w_{,2})_{,2}]\right\} * \delta w\,\mathrm{d}\Gamma = \int_\Gamma hf * \partial w\,\mathrm{d}\Gamma \quad \forall\, \delta w, \tag{33e}$$

$$\int_{\partial\Gamma_S} \frac{h^3}{12}\{[(2\mu+\lambda) \circledast w_{,11} + \lambda \circledast w_{,22}]n_1 + [2\mu \circledast w_{,21}]n_2\} * \delta w_{,1}\,\mathrm{d}\partial\Gamma_S \quad \forall\, \delta w_{,1} \tag{33f}$$

$$\int_{\partial\Gamma_S} \frac{h^3}{12}\{[2\mu \circledast w_{,12}]n_1 + [(2\mu+\lambda) \circledast w_{,22} + \lambda \circledast w_{,11}]n_2\} * \delta w_{,2}\,\mathrm{d}\partial\Gamma_S \quad \forall\, \delta w_{,2} \tag{33g}$$

$$\int_{\partial\Gamma_S}\left\{-\frac{h^3}{12}[[(2\mu+\lambda)\circledast w_{,11}+\lambda\circledast w_{,22}]_{,1}+[2\mu\circledast w_{,21}]_{,2}]n_1\right.$$

$$-\frac{h^3}{12}[[(2\mu+\lambda)\circledast w_{,22}+\lambda\circledast w_{,11}]_{,2}+[2\mu\circledast w_{,12}]_{,1}]n_2+h[\bar{S}_{11}w_{,1}+\bar{S}_{12}w_{,2}]n_1$$

$$\left.+h[\bar{S}_{21}w_{,1}+\bar{S}_{22}w_{,2}]n_2\right\}*\delta w\,\mathrm{d}\partial\Gamma_S \quad \forall\,\delta w. \tag{33h}$$

Only conditions (33e), (33h) are affected by terms related to $\bar{\mathbf{S}}$ while eqns (33a–d) furnish the classical thin plate equilibrium equations on the plane x_1–x_2 in the case of homogeneous data. Under suitable assumptions on μ and λ (thermodynamic compatibility), this second problem is well posed (Fabrizio and Lazzari, 1991) and it can be concluded that the perturbed motion is described by the function w only, while v_1 and v_2 are null.

The variations δw, $\delta w_{,1}$, $\delta w_{,2}$ cannot be independent of each other in the boundary. In particular, the derivative $\partial_n\delta w$ in the direction of the boundary normal and $\partial_t\delta w$ in the direction of the tangent, can be introduced by the following equations:

$$\delta w_{,1}=\partial_n\delta w\,n_1-\partial_t\delta w\,n_2, \tag{34a}$$

$$\delta w_{,2}=\partial_t\delta w\,n_1+\partial_n\delta w\,n_2. \tag{34b}$$

The substitution of the partial derivatives in the previous conditions permits deriving the problem in the following differential form:

$$\frac{h^3}{12}[\Delta(\lambda\circledast\Delta w)+(2\mu\circledast w_{,11})_{,11}+(4\mu\circledast w_{,12})_{,12}+(2\mu\circledast w_{,22})_{,22}]$$

$$+h[(\bar{S}_{11}w_{,1})_{,1}+(\bar{S}_{12}w_{,1})_{,2}+(\bar{S}_{12}w_{,2})_{,1}+(\bar{S}_{22}w_{,2})_{,2}]=hf \quad \text{on } \Gamma\times(0,T), \tag{35a}$$

$$(2\mu+\lambda)\circledast(w_{,11}n_1^2+w_{,22}n_2^2)+\lambda\circledast(w_{,11}n_2^2+w_{,22}n_1^2)+4\mu\circledast w_{,12}n_1n_2=0$$
$$\text{on } \partial\Gamma_S\times(0,T), \tag{35b}$$

$$-\frac{h^3}{12}[[(2\mu+\lambda)\circledast w_{,11}+\lambda\circledast w_{,22}]_{,1}+[2\mu\circledast w_{,21}]_{,2}]n_1-\frac{h^3}{12}[[(2\mu+\lambda)\circledast w_{,22}$$

$$+\lambda\circledast w_{,11}]_{,2}+[2\mu\circledast w_{,12}]_{,1}]n_2+h[\bar{S}_{11}w_{,1}+\bar{S}_{12}w_{,2}]n_1+h[\bar{S}_{21}w_{,1}+\bar{S}_{22}w_{,2}]n_2$$

$$-\frac{h^3}{12}\partial_t[2\mu\circledast[(w_{,22}-w_{,11})n_1n_2+w_{,12}(n_1^2-n_2^2)]]=0 \quad \text{on } \partial\Gamma_S\times(0,T), \tag{35c}$$

where the stress $\bar{\mathbf{S}}$ of the fundamental motion affects the field equation (35a) and the Neumann boundary condition related to Kirchhoff's shear force (35c).

The equations of the elastic case can be obtained if $\mu(t)$ and $\lambda(t)$ are constant in time. In this case the more usual stiffness parameters E (elastic modulus) and ν (Poisson coefficient) can be used, taking into account that, for the material constrained as previously, the two relations $E=4\mu/(\lambda+\mu)$ and $\nu=\lambda/(\lambda+2\mu)$, hold.

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